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The quantum oscillator on complex projective space (Lobachewski space) in a constant magnetic field and the issue of generic boundary conditions

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Abstract

We perform a one-parameter family of self-adjoint extensions characterized by the parameter ω_0. This allows us to get generic boundary conditions for the quantum oscillator on N-dimensional complex projective space ($\mathbb{C}P^N$) and on its non-compact version, i.e., Lobachewski space ($\mathcal{L}_N$) in the presence of a constant magnetic field. As a result, we get a family of energy spectra for the oscillator. In our formulation the already known result of this oscillator also belongs to the family. We have also obtained an energy spectrum which preserves all the symmetries (full-hidden symmetry and rotational symmetry) of the oscillator. The method of self-adjoint extensions has also been discussed for a conic oscillator in the presence of the constant magnetic field.

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1. Introduction

The quantum oscillator plays a fundamental role in theoretical physics due to its exact solvability and over-complete symmetry. The study of the oscillator became more interesting when the Euclidian oscillator was generalized on a curved space with a constant curvature by Higgs. This generalized oscillator which is known as the Higgs oscillator [1] for the obvious reason possesses lots of interesting features. For review see [2]. The Euclidian oscillator has also been generalized on a Kähler space, and the various properties of the system have been discussed in [3]. In [4], the exact solution of the quantum oscillator in N-dimensional complex projective space ($\mathbb{C}P^N$), Lobachewski space ($\mathcal{L}_N$) and related to cones in the presence of the constant magnetic field has been discussed. The relevance of this system to the higher dimensional quantum Hall effect makes it interesting. It has been shown that the inclusion of the constant magnetic field does not break any existing hidden symmetry of the oscillator and super-integrability and exact solvability.

But the solution of [4] has been presented in terms of a fixed boundary condition. As a consequence full symmetry of the energy spectrum has not been obtained. It is however possible to get generic boundary conditions for the oscillator by making self-adjoint extensions of the Hamiltonian of the system [4]. Consideration of the generic boundary conditions is not merely mathematical. It can be shown that the generic boundary conditions help us to get a complete set of spectra of the Hamiltonian under consideration. In [5], such an issue has been considered for the quantum-mechanical oscillator on a Kähler conifold in two dimensions, and it has been shown that the consideration of self-adjoint extensions can help us to get the energy spectrum, which is degenerate with respect to the orbital and azimuthal quantum numbers.

In our present work, we are going to address this issue for the oscillator defined on N-dimensional complex projective space ($\mathbb{C}P^N$) and on its non-compact version, i.e., Lobachewski space ($\mathcal{L}_N$) in the presence of a constant magnetic field. We will perform a one-parameter family of self-adjoint extensions [10] of the initial domain of the radial Hamiltonian of the harmonic oscillator [4] by the von Neumann method [10]. This will help us to construct generic boundary conditions. We will show that for a specific value of the self-adjoint extension parameter ω_0 we can recover the known result [4], and for other values of the extension parameter ω_0 we will get other energy spectra which were not known so far. We will also discuss about the degeneracy of the energy spectrum with respect to different quantum numbers, which has been possible for considering a one-parameter family of self-adjoint extensions of the radial Hamiltonian of the oscillator [4].

However, the importance of self-adjointness of a unitary operator is far fundamental. As we know, the evolution of a quantum system is dictated by a unitary group and the generator of that group is the Hamiltonian itself. According to Stone's theorem [10], generators of the unitary group (in this case Hamiltonian) should be self-adjoint. So, for a non-self-adjoint operator we should search for a self-adjoint extensions if possible. If the system has many self-adjoint extensions then different self-adjoint extensions should unveil different physics for the system.

The paper is organized as follows. In section 2, we discuss the quantum oscillator on complex projective space ($\mathbb{C}P^N$) and Lobachewski space ($\mathcal{L}_N$) in a background constant magnetic field. In section 3, we perform the self-adjoint extensions of the radial Hamiltonian of the oscillator discussed in the previous section, and we make some observations for some particular values of the extension parameter ω_0. Here, we show that it is possible to retain complete degeneracy in the energy spectrum (full-hidden symmetry and rotational symmetry). In section 4, the method of self-adjoint extensions has been discussed for the conic oscillator in the constant magnetic field. We conclude in section 5.

2. Quantum oscillator on $\mathbb{C}P^N$ and $\mathcal{L}_N$ with the background constant magnetic field

The quantum oscillator on complex projective space ($\mathbb{C}P^N$) and on Lobachewski space ($\mathcal{L}_N$) with the background constant magnetic field B is defined by the symplectic structure Ω and the Hamiltonian $\widehat{\mathcal{H}}$ respectively as

$$\Omega = \mathrm{d}\pi_a \wedge \mathrm{d}z_a + \mathrm{d}\bar{\pi}_a \wedge \mathrm{d}\bar{z}^a + \mathrm{i}Bg_{a\bar{b}}\,\mathrm{d}z^a \wedge \mathrm{d}\bar{z}^b \tag{2.1}$$

$$\widehat{\mathcal{H}} = \tfrac{1}{2}g^{a\bar{b}}(\hat{\pi}_a\hat{\bar{\pi}}_b + \hat{\bar{\pi}}_b\hat{\pi}_a) + \omega^2 g^{\bar{a}b}K_{\bar{a}}K_b, \tag{2.2}$$

where the metric is of the form

$$g^{\bar{a}b} = \frac{2}{r_0^2}(1+\epsilon z\bar{z})(\delta^{ab} + \epsilon z^a\bar{z}^b), \tag{2.3}$$

and the Kähler potential K and its derivatives K_a, $K_{\bar{a}}$ are given by

$$\begin{aligned} K &= \frac{r_0^2}{2\epsilon}\log(1+\epsilon z\bar{z}), & \epsilon &= \pm 1, \\ K_a &= \frac{\partial K}{\partial z^a} = \frac{r_0^2}{2}\frac{\bar{z}^a}{1+\epsilon z\bar{z}}, & K_{\bar{a}} &= \frac{\partial K}{\partial \bar{z}^a} = \frac{r_0^2}{2}\frac{z^a}{1+\epsilon z\bar{z}}. \end{aligned} \tag{2.4}$$

The representation of the momentum operators π_a and $\bar{\pi}_a$ consistent with the symplectic structure (2.1) takes the forms

$$\hat{\pi}_a = -\mathrm{i}\left(\hbar\partial_a + \frac{B}{2}K_a\right), \qquad \hat{\bar{\pi}}_a = -\mathrm{i}\left(\hbar\partial_{\bar{a}} - \frac{B}{2}K_{\bar{a}}\right). \tag{2.5}$$

In order to investigate the maximum possible energy spectra for the oscillator, let us consider the spectral problem

$$\widehat{\mathcal{H}}\Psi = E\Psi, \qquad \hat{J}_0\Psi = s\Psi, \qquad \hat{\mathbf{J}}^2\Psi = j(j+N-1)\Psi. \tag{2.6}$$

It is convenient to transform to the $2N$-dimensional spherical coordinates (r, ϕ_i), where $i = 1, \ldots, 2N-1$, r is a dimensionless radial coordinate taking values in the interval $[0, \infty)$ for $\epsilon = +1$, and in $[0, 1]$ for $\epsilon = -1$ and ϕ_is are appropriate angular coordinates. In this spherical coordinates the above energy eigenvalue equation in equation (2.6) can be separated into radial coordinate if we consider the trial wavefunction of the form

$$\Psi = \psi(r)D_s^j(\phi_i), \tag{2.7}$$

where $D_s^j(\phi_i)$ is the eigenfunction of the operators $\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_0$. It can be expressed via $2N$-dimensional Wigner functions, $D_s^j(\phi_i) = \sum_{m_i} c_{m_i} D^j_{m_i,s}(\phi_i)$, where j, m_i denote the total and azimuthal angular momentum quantum number, respectively,

$$\widehat{J}_0 D_s^j(\phi_i) = s D_s^j(\phi_i), \tag{2.8}$$

$$\widehat{\mathbf{J}}^2 D_s^j(\phi_i) = j(j+N-1)D_s^j(\phi_i), \qquad \widehat{J}_3 D^j_{m_i,s} = m_i D^j_{m_i,s}, \tag{2.9}$$

$$m, s = -j, -j+1, \ldots, j-1, j \qquad j = 0, 1/2, 1, \ldots. \tag{2.10}$$

Separating the differential equation we get the radial eigenvalue equation of the form

$$H(r)\psi(r) = E\psi(r), \tag{2.11}$$

where the radial Hamiltonian in equation (2.11) can be written in spherical coordinates as follows:

$$\begin{aligned} H(r) = -\frac{\hbar^2}{2r_0^2}(1+\epsilon r^2)\Bigg[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2N-1+\epsilon r^2}{r(1+\epsilon r^2)}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{4j(j+N-1)}{\epsilon r^2(1+\epsilon r^2)} \\ + \frac{\epsilon}{(1+\epsilon r^2)}\left(2s + \frac{\mu_B}{\epsilon}\right)^2 - \frac{\omega^2 r_0^4 r^2}{\hbar^2(1+\epsilon r^2)^2} + \frac{\epsilon\mu_B^2}{(1+\epsilon r^2)^2}\Bigg], \end{aligned} \tag{2.12}$$

where

$$r = \sqrt{z\bar{z}}, \qquad \mu_B = \frac{Br_0^2}{2\hbar}, \tag{2.13}$$

and we have replaced $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}}_0$ by their eigenvalues $j(j+N-1)$ and s, respectively.

We now move to the following section to discuss the self-adjointness of the radial Hamiltonian $H(r)$ of equation (2.12).

3. Self-adjointness of the radial Hamiltonian

The effective radial Hamiltonian $H(r)$ of equation (2.12) is formally self-adjoint, but formal self-adjointness does not mean that it is self-adjoint on a given domain [11]. This operator $H(r)$ belongs to an unbounded differential operator defined on a Hilbert space. As we have mentioned in our introduction, we will now perform self-adjoint extensions of the operator $H(r)$ by von Neumann's method [10]. But before that let us briefly review here the von Neumann method for the sake of completeness.

Let us consider an unbounded differential operator T defined over a Hilbert space $\mathcal{H}$ and consider a domain $D(T) \subset \mathcal{H}$ for the operator T such that it becomes symmetric on the domain $D(T) \subset \mathcal{H}$. Note that the operator T is called symmetric or Hermitian if $(T\phi, \chi) = (\phi, T\chi)\, \forall \phi, \chi \in D(T)$, where $(.,.)$ is the inner product defined over the Hilbert space $\mathcal{H}$. Let $D(T^\dagger)$ be the domain of the corresponding adjoint operator $T^\dagger$. The operator T is self-adjoint iff $T = T^\dagger$ and $D(T) = D(T^\dagger)$.

We now state the criteria of self-adjointness of a symmetric operator T according to the von Neumann method. We need to find out the deficiency subspaces (it is actually a null space) $D^\pm \equiv \mathrm{Ker}(\mathrm{i} \mp T\dagger)$ and the deficiency indices $n^\pm(T) \equiv \dim(D^\pm)$. Depending upon $n^\pm$, T is classified as [10]

(1) T is essentially self-adjoint, if $n^+ = n^- = 0$.
(2) T has an n^2-parameter(real) family of self-adjoint extensions, if $n^+ = n^- = n \neq 0$.
(3) T has no self-adjoint extensions, if $n^+ \neq n^-$. In this case T is called maximally symmetric.

We now return to the discussion of our effective radial differential operator $H(r)$. This operator is symmetric in the domain,

$$D(H(r)) = \{\phi(r) : \phi(r) = \phi'(r) = 0, \text{ absolutely continuous, square integrable over its full range with measure } \mathrm{d}\mu,\} \tag{3.1}$$

where $\mathrm{d}\mu = \frac{r^{2N-1}}{(1+\epsilon r^2)^{2N-1}}\,\mathrm{d}r$, $\phi'(r)$ is the derivative of $\phi(r)$ with respect to r. The domain of the adjoint operator $H^\dagger(r)$, whose differential expression is same as $H(r)$ due to formal self-adjointness, is given by

$$D^\dagger(H(r)) = \{\phi(r) : \text{absolutely continuous, square integrable on the half line with measure } \mathrm{d}\mu\}, \tag{3.2}$$

$H(r)$ is obviously not self-adjoint [10], because

$$D(H(r)) \neq D(H^\dagger(r)). \tag{3.3}$$

So we may ask whether there is any possible self-adjoint extensions [10] for the problem? To answer this question we need to investigate whether there is any square-integrable solutions for the differential equations

$$H(r)^\dagger \phi^\pm = \pm \mathrm{i}\phi^\pm. \tag{3.4}$$

The square-integrable solutions of equation (3.4) apart from normalization are given by

$$\phi^\pm = \begin{cases} Dt^{\frac{c-2}{2}}(1-t)^{\frac{b^\pm + a^\pm - c}{2}}\, {}_2F_1(a^\pm, b^\pm; c; t), & \text{for} \quad \epsilon = 1; \\ Dt^{\frac{c-2}{2}}(1-t)^{-\delta - 2a^\pm - \frac{c}{2}+1}\, {}_2F_1(a^\pm, b^\pm, c; t) & \text{for} \quad \epsilon = -1, \end{cases} \tag{3.5}$$

where the constants $a^\pm = a(\pm i)$, $b^\pm = b(\pm i)$ and c of the Hypergeometric function [12] ${}_2F_1(a^\pm, b^\pm, c; t)$ are given in the general form as

$$a(k) = \frac{1}{2}\left(2j + N + \epsilon\delta - \sqrt{\frac{2r_0^2 k}{\epsilon\hbar^2} + N^2 + \frac{\omega^2 r_0^4}{\hbar^2} + \mu_B^2}\right),$$
$$b(k) = \begin{cases} -a(k) + \delta + j_1 + 1, & \text{for} \quad \epsilon = 1; \\ a(k) + \delta, & \text{for} \quad \epsilon = -1; \end{cases} \tag{3.6}$$

$$c = j_1 + 1, \qquad j_1 = 2j + N - 1, \qquad \delta^2 = \frac{\omega^2 r_0^4}{\hbar^2} + \left(2s + \frac{\mu_B}{\epsilon}\right)^2,$$
$$t = \begin{cases} \dfrac{r^2}{1+r^2}, & \text{for} \quad \epsilon = 1; \\ r^2, & \text{for} \quad \epsilon = -1. \end{cases} \tag{3.7}$$

The existence of these complex eigenvalues of $H(r)^\dagger$ signifies that $H(r)$ is not self-adjoint. The solutions $\phi^\pm$ belong to the null space $D^\pm$ of $H(r)^\dagger \mp i$, where $D^\pm \in D^\dagger(H)$. The dimensions $n^\pm$ of $D^\pm$ are known as deficiency indices and are given by

$$n^\pm = \dim(D^\pm). \tag{3.8}$$

Since in our case the deficiency indices are $n^+ = n^- = 1$, we can get a one-parameter family of self-adjoint extensions of $H(r)$. The self-adjoint extensions of $H(r)$ are given by $H(r)^{\omega_0}$ with the domain $D(H(r)^{\omega_0})$, where

$$D(H(r)^{\omega_0}) = \{\psi(r) = \phi(r) + \phi^+(r) + \mathrm{e}^{\mathrm{i}\omega_0}\phi^-(r) : \phi(r) \in D(H(r)), \omega_0 \in \mathbb{R}(\mathrm{mod}\ 2\pi)\}. \tag{3.9}$$

The bound state solutions of $H(r)^{\omega_0}$ are of the form

$$\psi(r) = \begin{cases} Ct^{\frac{c-2}{2}}(1-t)^{\frac{b+a-c}{2}}\,{}_2F_1(a,b;c;t), & \text{for} \quad \epsilon = 1; \\ Ct^{\frac{c-2}{2}}(1-t)^{-\delta-2a-\frac{c}{2}+1}\,{}_2F_1(a,b,c;t), & \text{for} \quad \epsilon = -1, \end{cases} \tag{3.10}$$

where $a = a(E)$, $b = b(E)$, c and t are given in the general form in equations (3.6) and (3.7). C is the normalization constant. To find out the eigenvalues we have to match the function $\psi(r)$ with the domain equation (3.9) at $r \to 0$. In the limit $r \to 0$,

$$\psi(r) \to \begin{cases} Ct^{\frac{c-2}{2}}(1-t)^{\frac{b+a-c}{2}}[\Gamma_1(a,b,c) + (1-t)^{c-a-b}\Gamma_2(a,b,c)], & \text{for} \quad \epsilon = 1; \\ Ct^{\frac{c-2}{2}}[\Gamma_1(a,b,c) + (1-t)^{1+\frac{c}{2}}\Gamma_2(a,b,c)], & \text{for} \quad \epsilon = -1; \end{cases} \tag{3.11}$$

and

$$\phi^+(r) + \mathrm{e}^{\mathrm{i}\omega_0}\phi^-(r) \to \begin{cases} Dt^{\frac{c-2}{2}}(1-t)^{\frac{b+a-c}{2}}[\bar\Gamma_1 + (1-t)^{c-a-b}\bar\Gamma_2], & \text{for} \quad \epsilon = 1; \\ Dt^{\frac{c-2}{2}}[\bar\Gamma_1 + (1-t)^{1+\frac{c}{2}}\bar\Gamma_2], & \text{for} \quad \epsilon = -1; \end{cases} \tag{3.12}$$

where for any three constants m, n and p, $\Gamma(m, n, p)$ s are of the form

$$\Gamma_1(m,n,p) = \frac{\Gamma(p)\Gamma(p-m-n)\Gamma(m+n-p+1)\Gamma(1-p)}{\Gamma(p-m)\Gamma(p-n)\Gamma(n-p+1)\Gamma(m-p+1)},$$
$$\Gamma_2(m,n,p) = \frac{\Gamma(p)\Gamma(m+n-p)\Gamma(p-m-n+1)\Gamma(1-p)}{\Gamma(m)\Gamma(n)\Gamma(1-n)\Gamma(1-m)} \tag{3.13}$$

and

$$\bar\Gamma_1 = \Gamma_1(a^+,b^+,c) + \mathrm{e}^{\mathrm{i}\omega_0}\Gamma_1(a^-,b^-,c), \qquad \bar\Gamma_2 = \Gamma_2(a^+,b^+,c) + \mathrm{e}^{\mathrm{i}\omega_0}\Gamma_2(a^-,b^-,c) \tag{3.14}$$

Now comparing the respective coefficients in equations (3.11) and (3.12) we get the eigenvalue equation

$$f(E) \equiv \frac{\Gamma(a)\Gamma(b)\Gamma(1-b)\Gamma(1-a)}{\Gamma(c-a)\Gamma(c-b)\Gamma(b-c+1)\Gamma(a-c+1)} = \mathcal{M}\frac{\cos(\beta + \omega_0/2)}{\cos(\alpha + \omega_0/2)}, \tag{3.15}$$

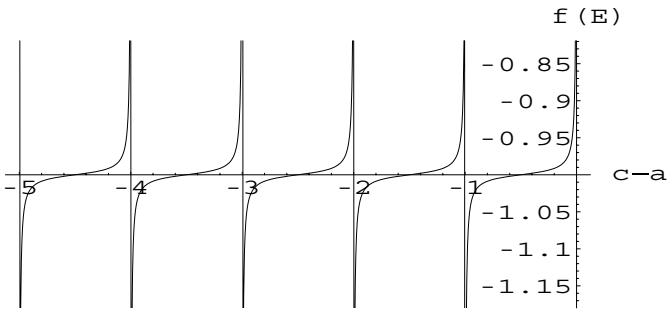

Figure 1. A plot of equation (3.15) using Mathematica with $N = 2, \epsilon = -1, j = 0$ (actually we have taken the limit $j \to 0$, so that equation (3.15) makes sense), $\delta = 0.001$ and energy range $c - a$ from -5 to 0. The horizontal axis labelled by $c - a$ corresponds to the rhs $= 0$ of equation (3.15).

where

$$\mathcal{M} = \frac{|\Gamma(c-a^{\pm})||\Gamma(c-b^{\pm})||\Gamma(b^{\pm}-c+1)||\Gamma(a^{\pm}-c+1)|}{|\Gamma(a^{\pm})||\Gamma(b^{\pm})||\Gamma(1-a^{\pm})||\Gamma(1-b^{\pm})|}, \tag{3.16}$$

$$\beta = |\arg(\Gamma(c-a^{\pm}))| + |\arg(\Gamma(c-b^{\pm}))| + |\arg(\Gamma(b^{\pm}-c+1))| + |\arg(\Gamma(a^{\pm}-c+1))|, \tag{3.17}$$

$$\alpha = |\arg(\Gamma(a^{\pm}))| + |\arg(\Gamma(b^{\pm}))| + |\arg(\Gamma(1-a^{\pm}))| + |\arg(\Gamma(1-b^{\pm}))|. \tag{3.18}$$

The eigenvalue for the general value of ω_0 can be calculated by plotting the graph of equation (3.15). We have plotted the graph of equation (3.15) in figure 1 and figure 2 for getting a complete understanding of the behaviour of the spectrum with respect to the self-adjoint extension parameter ω_0. But we can immediately calculate the eigenvalue analytically at least for some values of the extension parameter ω_0 in the boundary condition. So to appreciate constructing generalized boundary condition we now investigate some special cases.

3.1. Case 1

When the right-hand side of equation (3.15) is infinity, we get $a = \pm n$ or $b = \pm n$. $a = -n$ or $b = -n$ leads to the eigenvalue, already calculated in [4],

$$E_{n,j,s} = \frac{\epsilon\hbar^2}{2r_0^2}\left[(2n + 2j + N + \epsilon\delta)^2 - \left(\frac{\omega^2 r_0^4}{\hbar^2} + N^2 + \mu_B^2\right)\right]. \tag{3.19}$$

The radial quantum number is given by

$$n = \begin{cases} 0, 1, \ldots, \infty & \text{for} \quad \epsilon = 1 \\ 0, 1, \ldots, n^{\max} = [\delta/2 - j - 1] & \text{for} \quad \epsilon = -1. \end{cases} \tag{3.20}$$

For $a = +n$ and $b = +n$ the energy spectrum will be the same expression (3.19), with n replaced by $-n$.

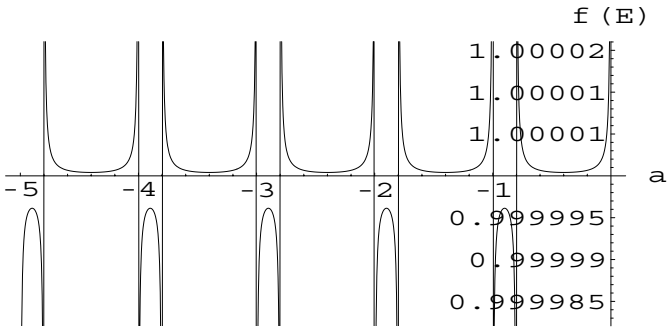

Figure 2. A plot of equation (3.15) using Mathematica with $N = 2, \epsilon = +1, j = 0$ (actually we have taken the limit $j \to 0$, so that equation (3.15) makes sense), $\delta = 1.2$ and energy range a from -5 to 0. The horizontal axis labelled by a corresponds to the rhs $\neq 0$ of equation (3.15).

3.2. Case 2

We can also make the right-hand side of equation (3.15) zero, which gives us $c - b = \pm n$ or $c - a = \pm n$. For $c - b = +n$, the energy spectrum becomes

$$E_{n,j,s} = \frac{\epsilon\hbar^2}{2r_0^2}\left[(2n - 2j - N + \delta)^2 - \left(\frac{\omega^2 r_0^4}{\hbar^2} + N^2 + \mu_B^2\right)\right], \tag{3.21}$$

for $c - b = -n$, n in (3.21) will be replaced by $-n$ and the radial quantum number n is given in (3.20). For $c - a = n$,

$$E_{n,j,s} = \frac{\epsilon\hbar^2}{2r_0^2}\left[(2n - 2j - N + \epsilon\delta)^2 - \left(\frac{\omega^2 r_0^4}{\hbar^2} + N^2 + \mu_B^2\right)\right]. \tag{3.22}$$

For $c - a = -n$, n in (3.22) will be replaced by $-n$ and the radial quantum number n is given in (3.20).

3.3. Case 3

For $c - b = +n + b$ and $c - a = +n + a$, we get degenerate (degenerate with respect to the orbital quantum number j) eigenvalue

$$E_{n,s} = \frac{\hbar^2}{2r_0^2}\left[(n + \delta)^2 - \left(\frac{\omega^2 r_0^4}{\hbar^2} + N^2 + \mu_B^2\right)\right], \qquad \text{for} \quad \epsilon = 1. \tag{3.23}$$

For $c - b = -n + b$ and $c - a = -n + a$, we get

$$E_{n,s} = -\frac{\hbar^2}{2r_0^2}\left[(n + \delta)^2 - \left(\frac{\omega^2 r_0^4}{\hbar^2} + N^2 + \mu_B^2\right)\right], \qquad \text{for} \quad \epsilon = -1. \tag{3.24}$$

3.4. Case 4

Even if, we can get the totally degenerate eigenvalue when $c - b = c - a \pm n$ and the form of the spectrum is given by

$$E_n = \frac{\hbar^2}{2r_0^2}\left[n^2 - \left(\frac{\omega^2 r_0^4}{\hbar^2} + N^2 + \mu_B^2\right)\right], \qquad \text{for} \quad \epsilon = +1. \tag{3.25}$$

For $a+b+c=\pm n$, we get

$$E_n = -\frac{\hbar^2}{2r_0^2}\left[n^2 - \left(\frac{\omega^2 r_0^4}{\hbar^2} + N^2 + \mu_B^2\right)\right], \qquad \text{for} \quad \epsilon = -1. \tag{3.26}$$

4. Self-adjointness of the conic oscillator in a constant magnetic field

The study of self-adjointness of the conic oscillator in a constant background magnetic field is just a straightforward generalization of what we have done so far. The $\nu-$ parametric family of cones over $\mathbb{C}P^N$ and $\mathcal{L}_N$ is defined by the Kähler potential

$$K = \frac{r_0^2}{2\epsilon}\log[1+\epsilon(z\bar{z})^\nu], \qquad \nu > 0; \qquad \epsilon = \pm 1. \tag{4.1}$$

The metric is given by

$$g_{a\bar{b}} = \frac{\nu r_0^2 (z\bar{z})^{\nu-1}}{2(1+\epsilon(z\bar{z})^\nu)}\left(\delta_{ab} - \frac{1-\nu+\epsilon(z\bar{z})^\nu}{z\bar{z}(1+\epsilon(z\bar{z})^\nu)}\bar{z}^a z^b\right). \tag{4.2}$$

The Hamiltonian of the system is the same as equation (2.2). After doing some algebra on the energy eigenvalue equation of equation (2.6), we can arrive at the radial Hamiltonian given by equation (2.12) with

$$\delta^2 = \frac{\omega^2 r_0^4}{\hbar^2} + \left(2\frac{s}{\nu} + \frac{B_0 r_0^2}{2\epsilon\hbar}\right)^2 \tag{4.3}$$

and

$$j_1^2 = \frac{(2j+N-1)^2}{\nu} + \frac{\nu-1}{\nu}\left[(N-1)^2 - \frac{4s^2}{\nu}\right]. \tag{4.4}$$

One can also perform the self-adjoint extension of the radial Hamiltonian of this system. The procedure is exactly the same as what we have done above. Note that the result will reduce to the result of [5] for the magnetic field $B=0$ and $N=2$.

5. Discussion

The issue of self-adjointness, as pointed out in the introduction, is of paramount importance in the quantum system due to Stone's theorem. It guarantees the spectrum to be the subset of the real line. Otherwise in principle the spectrum could be the subset of the complex plane. The complex eigenvalue could have importance in the dissipative system. However, in our work we have concentrated on bound states of the quantum oscillator on complex projective space ($\mathbb{C}P^N$) and Lobachewski space ($\mathcal{L}_N$) in the background constant magnetic field. So the Hamiltonian self-adjointness is must in our case.

We have obtained a generic boundary condition for the harmonic oscillator on $\mathbb{C}P^N(\mathcal{L}_N)$ in the constant magnetic field [4], and as a result we have obtained a ω_0-parameter family of an energy eigenvalue given by equation (3.15). There exists an energy spectrum at each point on the circle $e^{i\omega_0}$. We have shown that this generic boundary condition can restore the angular momentum degeneracy in the energy spectrum for a fixed value of the extension parameter ω_0. In subsection (3.3), we have obtained the eigenvalue which is independent of the orbital angular momentum quantum number j. In subsection (3.4), we have obtained the eigenvalue which is independent of both the orbital and azimuthal quantum numbers. For consistency checking, we have also recovered the result of [4] in subsection (3.1). Not only this, we have also shown that it allows us to obtain more solutions for different values of the

extension parameter ω_0; for example we have calculated a case in subsection (3.2). We have discussed the conic oscillator in a constant magnetic field background from the perspective of self-adjointness of the system. It is a straightforward extension of what we have done for the oscillator on $\mathbb{C}P^N(\mathcal{L}_N)$ in the constant magnetic field.

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